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Einstein's Relation between Diffusion Constant and Mobility for a Diffusion Model

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An Ornstein–Uhlenbeck process in a periodic potential in R^d is considered. It has been shown previously that this process satisfies a central limit theorem in the sense that, by rescaling space and time in a suitable way, the distribution of the process converges to that of a Wiener process with nonsingular diffusion matrix. Here a rigorous proof is given of a version of Einstein's formula for this model, relating the diffusion constant to the "mobility" of the system.

KEY WORDS: Diffusive behavior; Einstein relation; Ornstein–Uhlenbeck process; geometric ergodicity.

1. INTRODUCTION

Einstein[2] showed that, as a consequence of the molecular-kinetic theory of heat, microscopic particles suspended in a viscous liquid undergo an irregular motion which can be described by the diffusion equation

$$\frac{\partial f}{\partial t} = -\frac{D}{2} \Delta f$$

where $f(\cdot, t)$ is the density of the suspended particles at time t, Δ is the Laplacian, and D is a constant. For D he established the relation

$$D = 2kT\alpha \tag{1}$$

where k is the Boltzmann constant, T is the absolute temperature, and α is the "mobility" of the particles: if an external force K acts on a particle, it acquires a mean velocity $v(K)$ due to this force; by Stokes' law, the

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quotient $\alpha := v(K)/K$ is independent of K and equal to $(6\pi\eta r)^{-1}$, η being the viscosity of the liquid and r the radius of the suspended particles.

Recently, studies have been done aimed at a rigorous mathematical understanding of Einstein's formula (1).[3]

In this paper I give a rigorous proof of a version of (1) for the following process modeling the motion of a particle in a changing environment:

$$\begin{aligned} dV_t &= -\beta V_t\, dt - \Phi'(X_t)\, dt + \sigma\, dW_t \\ dX_t &= V_t\, dt \end{aligned} \tag{2}$$

Here W_t denotes a standard Wiener process in R, Φ is a periodic function in $C^5(R)$, and β and σ are positive constants.

The model (2) describes an Ornstein–Uhlenbeck process in R with an additional drift of the velocity coming from a periodic potential Φ depending on the space coordinate. This model is understood as an approximation of the more complicated situation where the periodic Φ is replaced by an ergodic random potential Φ representing the "random environment" of the particle.

Remark. I here restrict consideration to the case (2) of one-dimensional V and X to keep the notation simple. One easily checks that the same proofs work in higher finite dimensions.

The model (2) has a unique solution (V_t, X_t) for each given measurable (V_0, X_0), (V_0, X_0) independent of $(W_t)_{t \in R^+}$.[1]

Since Φ is periodic, the process (V_t, X_t) can be considered as a process with state space $E := R \times \text{Tor}$, where Tor is the one-dimensional torus with the length of the period of Φ. The invariant measure of (V_t, X_t) on E is

$$\pi(dv, dx) := C^{-1} \exp\left[-2 \frac{\beta}{\sigma^2} \Phi(x) - \frac{\beta}{\sigma^2} v^2 \right] dv\, dx$$

with

$$C := \sigma \left(\frac{\pi}{\beta}\right)^{1/2} \int_{\text{Tor}} \exp\left[-2 \frac{\beta}{\sigma^2} \Phi(x) \right] dx$$

In the situation of our model, we will give meaning to Einstein's formula in the following way. In ref. 5 it has been shown that the process (2) has "diffusive behavior" in the sense that it satisfies the following central limit theorem (the corresponding result for ergodic random Φ has been obtained by Papanicolaou and Varadhan[4]):

Theorem 1. Let (V_t, X_t) be the solution of (2) for given arbitrary

(V_0, X_0). For $\varepsilon > 0$ define the process X^ε by $X^\varepsilon(t) = \varepsilon X(\varepsilon^{-2} t)$. Then for $\varepsilon \to 0$, the distribution of X^ε in $C[0, \infty)$ converges weakly to the distribution of a Wiener process with diffusion coefficient

$$D := 2 \int_E \int_0^\infty V P_t V \, dt \, d\pi \tag{3}$$

Here P_t denotes the transition semigroup of the process (2) and the functional $V\colon E \to R$ is given by $V(v, x) = v$.

The diffusion coefficient of the limit Wiener process given by this theorem will be taken as the "diffusion constant" in Einstein's formula. The "mobility" of our system will be defined as follows. In our framework, the motion (V_t^e, X_t^e) of a particle which is driven through the medium by an external force $e \in R$ is modeled by

$$\begin{aligned} dV_t^e &= -\beta V_t^e \, dt - \Phi'(X_t^e) \, dt + \sigma \, dW_t + e \, dt \\ dX_t^e &= V_t^e \, dt \end{aligned} \tag{4}$$

(I assume the mass of the particle to be 1). As we shall see in Section 2, for each $e \in R$, (V_t^e, X_t^e) has a unique invariant probability measure π_e on E. So the "mean velocity" $\bar{v}_e$ of the particle driven by the force e can be defined as

$$\bar{v}_e := \int_E v \, d\pi_e$$

The mobility is defined by

$$\alpha := \frac{d}{de} \bar{v}_e \bigg|_{e=0} \tag{5}$$

The main result of this paper can now be stated as follows:

Theorem 2. The mobility α is well defined by formula (5) and

$$D = \sigma^2 \beta^{-1} \alpha$$

We note that the quotient $D/\alpha = \sigma^2 \beta^{-1}$ given by this theorem equals $4 \int_E \frac{1}{2} v^2 \, d\pi$, which can be understood as four times the mean kinetic energy of a particle, in accordance with Einstein's formula (1).

The proof of the theorem is given in Sections 2 and 3. In Section 2 we show that the processes $(V_t^e, X^e{}_t)$ are uniformly geometrically ergodic for e in a neighborhood of 0. This result is used in Section 3, where we follow up a perturbational approach to establish the differentiability of π_e and $\bar{v}_e$ in e and to derive an explicit expression for α.

The idea of this derivation is roughly as follows. If π_e has a smooth density f_e which is differentiable in e (with smooth derivative F: $E \to R$), then f_e can be written in the form

$$f_e = f_0 + eF + r_e, \qquad r_e = o(e) \tag{6}$$

and solves

$$\left(L_0^* - e\frac{\partial}{\partial v}\right)(f_0 + eF + r_e) = 0 \tag{7}$$

where f_0 is the density of π_0 and L_0^* is the adjoint of the generator of the process (2) [notice that $L_e^* = L_0^* - e\,\partial/\partial v$ is the adjoint of the generator of (4)]. We therefore get the hierarchy of equations

$$L_0^* f_0 = 0 \tag{8}$$

$$L_0^* F = \frac{\partial}{\partial v} f_0 = \frac{-2\beta}{\sigma^2} V f_0 \tag{9}$$

$$\left(L_0^* - e\frac{\partial}{\partial v}\right) r_e = e^2 \frac{\partial}{\partial v} F \tag{10}$$

Equation (9) is solved by

$$F(v, x) = -\frac{2\beta}{\sigma^2} f_0 G(-v, x)$$

with

$$G = \int_0^\infty P_t V\,dt \qquad \text{(formally, } G = L_0^{-1} V\text{)}$$

[using the reversibility property $L_0^*(f_0\hat{\psi}) = f_0 L_0 \psi$, where $\hat{\psi}(v, x) = \psi(-v, x)$ for $\psi: E \to R$].

Hence,

$$\begin{aligned}
\alpha &= \frac{d}{de}\int_E V\,d\pi_e \bigg|_{e=0} \\
&= \int_E V \cdot F\,dv\,dx \\
&= \frac{2\beta}{\sigma^2}\int_0^\infty \int_E V \cdot P_t V\,d\pi\,dt \\
&= \frac{\beta}{\sigma^2} D
\end{aligned}$$

The main task will be to clarify in which sense these equations are meaningful; in particular, it has to be shown that the entities defined exist in a suitable sense. For this purpose we need the ergodicity results of Section 2, which are established using pure probabilistic methods.

I finish this section by introducing some more notation. By L_e I denote the generator of the process (4), i.e.,

$$L_e = v \frac{\partial}{\partial x} + [-\beta v - \Phi'(x) + e] \frac{\partial}{\partial v} + \frac{\sigma^2}{2} \frac{\partial^2}{\partial v^2}$$

P_t^e denotes the corresponding semigroup. For P_t^0 I also write P_t. $\delta_{\bar{x}}$ for $\bar{x} \in E$ is the Dirac measure in $\bar{x}$. $V: E \to R$ is defined by $V(v, x) = v$. Leb is the Lebesgue measure on R^2 or E, $|\bar{x}|$ the Euclidean norm of $\bar{x} \in R^2$, $\nabla_{\bar{x}} F$ the $\bar{x}$ gradient of a function $F(\bar{x},...)$, $\bar{x} \in R^l$. I use letters with bars $\bar{x}$, $\bar{y}$,... to denote pairs $(v, x) \in E$ or $(v, x) \in R^2$ and write $\bar{X}_t^e$ for (V_t^e, X_t^e) and $\bar{X}_t$ for $\bar{X}_t^0 = (V_t^0, X_t^0)$.

If not indicated otherwise, I consider $\bar{X}_t^e$ as a process in E (rather than R^2).

2. THE ERGODIC BEHAVIOR OF THE PROCESSES $\bar{X}_t^e$

In this section I show that the processes $\bar{X}_t^e$ are geometrically ergodic uniformly for e in a neighborhood of 0:

Theorem 3. The processes $\bar{X}_t^e$ have unique invariant probability measures π_e on E. For arbitrary $A > 0$ and bounded $B \subseteq E$, there exists $q < 1$ such that

$$\tfrac{1}{2} \|\delta_{\bar{x}} P_t^e - \pi_e\| \leqslant q^t \qquad (t > 0, \bar{x} \in B, |e| < A)$$

Proof. The idea of the proof is that the processes $\bar{X}_t^e$ behave almost like processes on a compact state space since, because of the friction term $-V_t^e\, dt$ in (4), states at high velocities can be well controlled.

Fix $A > 0$ and $B \subseteq E$ bounded. Choose $v_0 > 2^{1/2}(\sup_{x \in R} |\Phi'(x)| - A)\beta^{-1}$ large enough so that

$$K := \{((v, x), (u, y)) \in E^2 \mid |(v, u)| \leqslant v_0\} \tag{11}$$

contains B^2. The set K is used as a compact approximation of E^2; the condition on v_0 will be used to ensure that, due to the driving forces, a pair of particles will never stay for too long outside K. Let

$$^{\bar{x}, \bar{y}}\eta_t^e := \tfrac{1}{2} \|\delta_{\bar{x}} P_t^e - \delta_{\bar{y}} P_t^e\| \qquad (\bar{x}, \bar{y} \in E)$$

and

$$\eta_t^e := \sup_{(\bar{x}, \bar{y}) \in K} {}^{\bar{x}, \bar{y}}\eta_t^e \tag{12}$$

It is easily shown that $\eta_t^e \leqslant 1$ and η_t^e is nonincreasing in t.

Let $(\bar{x}, \bar{y}) \in K$, $|e| < A$, and take any coupling $(\bar{X}_t^e, \bar{Y}_t^e)$ of two processes (4) starting in $\bar{x}$, $\bar{y}$, respectively. Let

$$\tilde{\tau} := \inf\{t \geqslant 1 \mid (\bar{X}_t^e, \bar{Y}_t^e) \in K\}$$

and let ν be the distribution of $\tilde{\tau}$ on $[1, \infty)$. We then get, using the semigroup properties of P_t^e, the estimate

$${}^{\bar{x}, \bar{y}}\eta_t^e \leqslant \int_{[1,t]} \nu(ds)\eta_{t-s}^e \qquad (t > 1)$$

and, in fact, by a coupling argument,

$${}^{\bar{x}, \bar{y}}\eta_t^e \leqslant [\nu(\{1\}) - \delta]\eta_{t-1}^e + \int_{(1,t]} \nu(ds)\eta_{t-s}^e \tag{13}$$

where

$$\delta := \nu(\{1\}) - \tfrac{1}{2}\|\mu_1 - \mu_2\| \tag{14}$$

with μ_1 the distribution of $\bar{X}_1 1_K(\bar{X}_1, \bar{Y}_1)$ on E, and μ_2 the distribution of $\bar{Y}_1 1_K(\bar{X}_1, \bar{Y}_1)$ on E.

Notice that δ and ν depend on $\bar{x}$, $\bar{y}$, and e; we will use the notation $\delta(\bar{x}, \bar{y}; e)$, $\nu(\bar{x}, \bar{y}; e)$ to make this dependence explicit.

Our aim now is to show that, for suitable coupling of $\bar{X}_t$, $\bar{Y}_t$,

$$\inf_{\substack{e \in [-A, A] \\ \bar{x}, \bar{y} \in K}} \delta(\bar{x}, \bar{y}; e) > 0$$

and that there is a probability measure $\hat{\nu}$ on $[1, \infty)$ which is stochastically larger than ν for all $(\bar{x}, \bar{y}) \in K$, $e \in [-A, A]$. It will then be possible to derive from inequality (13) uniform exponential decay of ${}^{\bar{x}, \bar{y}}\eta_t^e$ in $\bar{x}, \bar{y} \in K$, $|e| \leqslant A$. I choose the *independent coupling* of $(\bar{X}_t^e, \bar{Y}_t^e)$.

The following rather technical proposition gives an estimate of the distribution of $(\bar{X}_t^e, \bar{Y}_t^e)$ for large velocities and of the distribution of $\tilde{\tau}$, by comparing with a simpler process, using the fact that for large v, the drift of V_t^e in (v, x) is governed by the term $-\beta v$.

Proposition 4(a). Let

$$c := \sup_{x \in R} |\Phi'(x)|$$

and let $\bar{v} > c\beta^{-1}$.

(i) Let τ_e be the stopping time

$$\tau_e := \inf\{t \geqslant 0 \mid |V_t^e| \leqslant \bar{v}\}$$

Then for all sufficiently small $\lambda \geqslant 0$, namely $0 \leqslant \lambda < (\beta\bar{v} - c)^2 (2\sigma^2)^{-2}$, there exists $\psi(\lambda) < \infty$ such that

$$E_{(v,x)} \exp(\lambda\tau_e) \leqslant \exp[\psi(\lambda)(|v| - \bar{v})]$$

for all $(v, x) \in E$ with $|v| \geqslant \bar{v}$, for all sufficiently small $|e| \in R$, namely $|e| \leqslant \beta\bar{v} - c$.

(ii) Let $m_{\bar{v}}$ be the measure on R_+ defined by the density $F_{\bar{v}}$:

$$F_{\bar{v}}(u) = \begin{cases} 0 & \text{if} \quad u \leqslant \bar{v} \\ 2a\sigma^{-2} \exp[-2a\sigma^{-2}(u - \bar{v})] & \text{if} \quad u > \bar{v} \end{cases} \tag{15}$$

with $a < \beta\bar{v} - c$. Let $\prec$ denote stochastic ordering of probability measures on R. Then the following holds: for all sufficiently small $|e|$, namely $|e| \leqslant \beta\bar{v} - c - a$, if the distribution of $|V_t^e|$ at time $t = 0$ is $\prec m_{\bar{v}}$ (for example, if $|V_0^e| \leqslant \bar{v}$ a.s.), then for all $t > 0$ the distribution of $|V_t^e|$ is $\prec m_{\bar{v}}$.

(b) Let $\bar{v} > 2^{1/2} c\beta^{-1}$. Consider two independent processes (4), $\bar{X}_t^e = (V_t^e, X_t^e)$ and $(\bar{Y}_t^e) = (U_t^e, Y_t^e)$. Then:

(i) For the stopping time

$$\tau_e := \inf\{t \geqslant 0 \mid |(V_t^e, U_t^e)| \leqslant \bar{v}\}$$

for all $0 \leqslant \lambda < (\beta\bar{v} - 2^{1/2}c)^2 (2\sigma^2)^{-1}$, there exists $\psi(\lambda) < \infty$ such that

$$E_{((v,x),(u,v))} \exp(\lambda\tau_e) \leqslant \exp[\psi(\lambda)(|(v,u)| - \bar{v})]$$

for all (v, x), $(u, y) \in E$ with $|(v, u)| \geqslant \bar{v}$; for all sufficiently small $|e|$, namely $|e| \leqslant \beta\bar{v}/\sqrt{2} - c$.

(ii) If the distribution of (V_t^e, U_t^e) at $t = 0$ is $\prec m_{\bar{v}}$, $m_{\bar{v}}$ being defined as in (15) with $a < \beta\bar{v} - 2^{1/2}c$, then for all $t > 0$, the distribution of $|(V_t^e, U_t^e)|$ is $\prec m_{\bar{v}}$, provided $|e|$ is small enough: $|e| < (\beta\bar{v} - a)/\sqrt{2} - c$.

Proof of (b) [the proof of (a) is similar and somewhat simpler]. I give a description of the idea, omitting the technical details of the proof. Let $\bar{v} > 2^{1/2} c\beta^{-1}$ be given. For $e \in R$ consider two independent processes (4), $\bar{X}_t^e = (X_t^e, V_t^e)$ and $\bar{Y}_t^e = (Y_t^e, U_t^e)$. Let h: $R^4 \to R_+$, $h((v, x), (u, y)) = (v^2 + u^2)^{1/2}$, i.e., the Euclidean norm of (v, u). The process $h(\bar{X}_t^e, \bar{Y}_t^e)$ behaves, according to Ito's lemma, like a diffusion in R with diffusion constant σ (since the driving Wiener processes of $\bar{X}_t^e$, $\bar{Y}_t^e$ have been chosen independent) as long as $h(\bar{X}_t^e, \bar{Y}_t^e) \neq 0$, and for $h \geqslant \bar{v}$ the drift (which is negative) has absolute value $\geqslant \beta h - 2^{1/2}(|e| + c) \geqslant \beta \bar{v} - 2^{1/2}(|e| + c) > a$ (provided $|e|$ is small enough). So a comparison of the process $h(\bar{X}_t^e, \bar{Y}_t^e)$ with the process in R,

$$dZ_t = -g(Z_t)\,dt + \sigma\,dW, \qquad g(z) = \begin{cases} a, & z \geqslant 0 \\ -a, & z < 0 \end{cases}$$

(which can be made precise by a coupling argument) shows

$$E_{(\bar{x}, \bar{y})} \exp(\lambda \tau) \leqslant E_{h(\bar{x}, \bar{y}) - \bar{v}} \exp(\lambda \eta) \qquad [h(\bar{x}, \bar{y}) \geqslant \bar{v}, \lambda \geqslant 0]$$

where

$$\eta := \inf\{t \geqslant 0 \mid Z_t = 0\}$$

Similarly, if the distribution of $h(\bar{X}_t^e, \bar{Y}_t^e) - \bar{v}$ at time $t = 0$ is $\prec$ the distribution of $|Z_t|$ at $t = 0$, then for all $t > 0$ the distribution of $h(\bar{X}_t^e, \bar{Y}_t^e) - \bar{v}$ is $\prec$ the distribution of $|Z_t|$ (provided $|e|$ is small enough). So the assertion of Proposition 4b follows by noting that (i) for $\lambda < a^2(2\sigma^2)^{-1}$ there exists $\psi(\lambda) > 0$ such that

$$E_z \exp(\lambda \eta) = \exp[\psi(\lambda) z] \qquad (z \geqslant 0)$$

(see ref. 4) and (ii) if the distribution of $|Z_0|$ is $\prec$ the invariant probability measure of $|Z_t|$,

$$m(dv) := 2a\sigma^{-2} \exp(-2a\sigma^{-2} v)\,dv \qquad (v \geqslant 0)$$

then for all $t > 0$ the distribution of $|Z_t|$ is $\prec m$ (which is also shown by a coupling argument).

By combining Proposition 4(b)(i) and (ii) with $\bar{v} = v_0$, it follows that for the stopping time $\tilde{\tau} = \inf\{t \geqslant 1 \mid (\bar{X}_t^e, \bar{Y}_t^e) \in K\}$, K given by (11) for $(\bar{x}, \bar{y}) \in K$, $|e| < A$, one has [here $M_{(\bar{x}, \bar{y})}$ is the distribution of $(\bar{X}_t^e, \bar{Y}_t^e)$ at $t = 1$ with $\bar{X}_0^e = \bar{x}$, $\bar{Y}_0^e = \bar{y}$]

$$\begin{aligned} E_{(\bar{x}, \bar{y})} \exp(\lambda \tilde{\tau}) &= \int_{E^2} E_{(\bar{w}, \bar{z})} \exp(\lambda \tilde{\tau})\, dM_{(\bar{x}, \bar{y})} \\ &\leqslant L := \int_0^\infty \exp[\psi(\lambda) v]\, 2a\sigma^{-2} \exp(-2a\sigma^{-2} v)\, dv \\ &< \infty \end{aligned}$$

[with $a = v_0 - 2^{1/2}(A + c)\beta^{-1}$] for sufficiently small λ [we can assume that in Proposition 4(b)(i), $\psi(\lambda) \to 0$ as $\lambda \to 0$ by the dominated convergence theorem of measure theory]. Consequently, $\tilde{\tau}$ is stochastically smaller than the probability measure $\hat{\nu}$ on $[1, \infty)$ with density

$$S(t)=\begin{cases}\lambda L\exp(-\lambda t), & t\geqslant t_0:=\ln(L)\lambda^{-1}\\ 0, & t<t_0\end{cases}$$

for all $|e|\leqslant A$ and starting points of $(\bar{X}_t^e, \bar{Y}_t^e)$, $(\bar{x}, \bar{y})\in K$.

To give an estimate of δ, we use the following result.

Proposition 5. For any $t>0$, $\bar{x}\in R^2$, and $e\in R$, $\delta_{\bar{x}}P_t^e$ has a density with respect to the Lebesgue measure. With respect to the variational norm, $\delta_{\bar{x}}P_t^e$ depends continuously on e and $\bar{x}$. Moreover, $\delta_{\bar{y}}P_t^e$ is three times differentiable in $\bar{y}$ in the sense that for any $t>0$, for sufficiently small $\Delta\bar{x}\in R^2$, $\delta_{\bar{x}+\Delta\bar{x}}P_t^e$ has a density with respect to $\delta_{\bar{x}}P_t^e$ which is in $L_2(\delta_{\bar{x}}P_t^e)$ and there are functions $J_1(\bar{x})$: $E\to R^2$, $J_2(\bar{x})$: $E\to R^4$, $J_3(\bar{x})$: $E\to R^8$ in $L_2(\delta_{\bar{x}}P_t^e)$ such that

$$\begin{aligned}&d(\delta_{\bar{x}+\Delta\bar{x}}P_t^e)/d(\delta_{\bar{x}}P_t^e)\\ &\quad=1+J_1(\bar{x})\,\Delta\bar{x}+\tfrac{1}{2}J_2(\bar{x})\,\Delta\bar{x}^2\\ &\qquad+\tfrac{1}{6}J_3(\bar{x})\,\Delta\bar{x}^3+|\Delta\bar{x}|^3 r(\Delta\bar{x}),\qquad r(\Delta\bar{x})\to 0\ \text{as}\ \Delta\bar{x}\to 0\end{aligned}$$

holds in $L_2(\delta_{\bar{x}}P_t^e)$. [Here $\Delta\bar{x}^i$ is to be understood as an element of R^{2^i} (componentwise multiplication) and $J_i\Delta\bar{x}$ as scalar product.] $\|J_1(\bar{x})\|_{L_2(\delta_{\bar{x}}P_t^e)}$ is bounded in $\bar{x}\in E$.

The proof of Proposition 5 is based on the following result.

Theorem 6 (Cameron–Martin–Girsanov formula, cf. ref. 6, Theorems 6.4.2, 8.1.1). Let $l\in N$, δ: $[0,\infty)\times R^l\to R^l$, γ: $[0,\infty)\times R^l\to R^l$, and α: $[0,\infty)\times R^l\to S_l$, where S_l is the set of symmetric, nonnegative-definite, real $l\times l$ matrices, be bounded and measurable. Let P be the distribution on $C([0,\infty), R^l)$ of the process

$$dZ_t=\delta(t,Z_t)\,dt+\alpha(t,Z_t)\,d\mathscr{W}_t,\qquad Z_0=x\in R^l \tag{16}$$

($\mathscr{W}$ is the l-dimensional standard Wiener process), and Q the distribution of the process

$$dZ_t=(\delta+\alpha\mu)(t,Z_t)\,dt+\alpha(t,Z_t)\,d\mathscr{W}_t,\qquad Z_0=x \tag{17}$$

Then for all $t>0$, $Q\ll P$ with respect to the σ-algebra $\mathscr{F}_t$ on $C[0,\infty)$ which is generated by the mappings $C[0,\infty)\to R^d$: $g\to g(s)$, $0\leqslant s\leqslant t$, and

$$dQ/dP(g)=\exp\left(\int_0^t\langle\gamma(u,g(u)),d\bar{g}(u)\rangle-\frac{1}{2}\int_0^t\langle\gamma(u,g(u)),\alpha\gamma(u,g(u))\rangle\,du\right)$$

where $\bar{g}(t)=g(t)-\int_0^t\delta(u,g(u))\,du$.

Remark 7. The boundedness assumption on $\delta, \gamma,$ and α in Theorem 6 can be dropped if δ, γ, and α are continuous and satisfy the following condition ensuring the existence of a solution of (16), (17) (see ref. 1, Satz 6): $\delta(t, \cdot)$, $\alpha(t, \cdot)$, $(\delta + \gamma\alpha)(t, \cdot)$ are Lipschitz continuous uniformly in $t \geqslant 0$. This follows from Theorem 6 by approximating δ, γ, and α by bounded functions δ_n, γ_n, and α_n $(n \in N)$ which coincide with δ, γ, and α on $[0, \infty) \times \{x \in R^l \mid |x| \leqslant n\}$.

Proof of Proposition 5. (I here consider the process $\bar{X}_t^e$ as a process in R^2 rather than E and prove the corresponding statement of Proposition 5 for this case; it then carries easily over to E as phase space, so I will not make any distinction in notation.)

(i) Existence of a Density of $\delta_{\bar{x}} P_t^e$. Theorem 6 is applied as follows: P is the distribution of (4) on $C([0, t], R^2)$ with $\bar{X}_t^e(0) = \bar{x}$; Q is the distribution of the process

$$d\tilde{X}_t = \tilde{V}_t \, dt, \qquad d\tilde{V}_t = \sigma \, dW_t, \qquad (\tilde{V}_0, \tilde{X}_0) = \bar{x}$$

The result then follows by projecting P and Q down onto the t coordinate and noting that the projection of Q [i.e., the distribution of (V_t, X_t) in R^2] has a density with respect to the Lebesgue measure.

(ii) Continuity. Theorem 6 is applied to the processes $\bar{X}_t^e$ and $\bar{X}_t^{e+a}$, $a \in R$, where $a \to 0$.

(iii) Differentiability. I show differentiability of first order; differentiability up to third order is proved in a similar way.

For any $\bar{x} \in R^2$, denote by ${}^{\bar{x}}\bar{X}_s^e$ the solution of (4) starting in $\bar{x}$. Fix $t > 0$. We will first keep $\bar{x}$ and $\Delta\bar{x}$ fixed and compare the processes

$$\bar{X}_s = (V_s, X_s) := {}^{\bar{x}}\bar{X}_s^e \qquad \text{and} \qquad \bar{X}'_s = (V'_s, X'_s) := {}^{\bar{x} + \Delta\bar{x}}\bar{X}_s^e$$

We use a transformation of $\bar{X}'_s$ to make Theorem 6 applicable. Take $F: R^2 \times R \to R$ four times continuously differentiable such that $F(0, \cdot) = 0$ and for each $\Delta\bar{x} = (\Delta v, \Delta x) \in R^2$,

$$F(\Delta\bar{x}, 0) = \Delta v, \qquad F(\Delta\bar{x}, t) = 0, \qquad \int_0^t F(\Delta\bar{x}, s) \, ds = -\Delta x$$

Define the process $\tilde{\bar{X}}_s = (\tilde{V}_s, \tilde{X}_s)$ by

$$\tilde{V}_s = V'_s - F(\Delta\bar{x}, s)$$

$$\tilde{X}_s = X'_s - \int_0^s F(\Delta\bar{x}, r) \, dr - \Delta x$$

Then $\tilde{\bar{X}}_0 = \bar{X}_0 = \bar{x}$ and $\tilde{\bar{X}}_t = \bar{X}'_t$. By Ito's formula, $(\tilde{V}_s, \tilde{X}_s)$ satisfies

$$d\tilde{X}_s = \tilde{V}_s\,ds$$

$$d\tilde{V}_s = \frac{\partial}{\partial s} F(\Delta\bar{x}, s)\,ds - \beta(V_s + F(\Delta\bar{x}, s))\,ds + \Phi'\left(\tilde{X}_s + \int_0^s F(\Delta\bar{x}, r)\,dr + \Delta x\right) ds + \sigma\,dW$$

Let

$$G(s, z, \Delta\bar{x}) := \frac{\partial}{\partial s} F(\Delta\bar{x}, s) - \beta F(\Delta\bar{x}, s) + \Phi'(z) - \Phi'\left(z - \int_0^s F(\Delta\bar{x}, r)\,dr + \Delta x\right)$$

that is, the difference of the v drift of the processes $(\tilde{V}_s, \tilde{X}_s)$, (V_s, X_s) at $(s, (w, z)) \in R_+ \times R^2$. (Notice that these two processes both start at $x \in R^2$.) Let P and Q be the distributions of $(\tilde{V}_s, \tilde{X}_s)$, (V_s, X_s), respectively, on $C([0, t], R^2)$ (the space of paths up to time t with the σ-algebra generated by the projections on the s coordinate, $0 \leqslant s \leqslant t$). By Theorem 6 and Remark 7, $Q \ll P$ and

$$\frac{dQ}{dP}[(v_s, x_s)_{s \leqslant t}] = \exp\left[\int_0^t G(s, x_s, \Delta\bar{x})\sigma^{-1}\,d\bar{v}_s - \frac{1}{2}\int_0^t G^2(s, x_s, \Delta\bar{x})\sigma^{-1}\,ds\right]$$

where $\bar{v}_s = v_s + \int_0^t \beta v_r + \Phi'(x_r)\,dr$ {which is a Brownian motion on $C([0, t], R^2)$ with respect to P and the canonical filtration}. As $G(s, x_s, \Delta\bar{x})$ is continuously differentiable in s for P-almost all $(v_s, x_s)_{s \leqslant t}$ [with derivative

$$\frac{d}{ds} G(s, x_s, \Delta\bar{x}) = [\frac{\partial}{\partial s} G(s, x_s, \Delta\bar{x}) + \frac{\partial}{\partial z} G(s, z, \Delta\bar{x})\bigg|_{z = x_s} \cdot v_s]$$

this expression can be rewritten as (see ref. 1, Corollaries 4.5.10–4.5.12)

$$\frac{dQ}{dP}[(v_s, x_s)_{s \leqslant t}] = \exp\Bigg[G(t, x_t, \Delta\bar{x})\sigma^{-1}\bar{v}_t - G(0, x_0, \Delta\bar{x})\sigma^{-1}\bar{v}_0 - \int_0^t \frac{d}{ds} G(s, x_s, \Delta\bar{x})\sigma^{-1}\bar{v}_s\,ds - \frac{1}{2}\int_0^t G^2(s, x_s, \Delta\bar{x})\sigma^{-1}\,ds\Bigg]$$

From this we see that $dQ(\Delta\bar{x})/dP$ [we write $Q(\Delta\bar{x})$ to indicate the dependence of Q on $\Delta\bar{x}$] is pointwise differentiable in $\Delta\bar{x}$ ($\bar{x}$ still being fixed) with derivative at $\Delta\bar{x}=0$,

$$\begin{aligned}\nabla_{\Delta\bar{x}} \frac{dQ(\Delta\bar{x})}{dP}\left[(v_s, x_s)_{s\leqslant t}\right]\bigg|_{\bar{x}=0} &= \nabla_{\Delta\bar{x}} G(t, x_t, 0)\sigma^{-1}\bar{v}_t - \nabla_{\Delta\bar{x}} G(0, x_0, 0)\sigma^{-1}\bar{v}_0 \\ &\quad - \int_0^t \nabla_{\Delta\bar{x}} \frac{d}{ds} G(s, x_s, 0)\sigma^{-1}\bar{v}_s\, ds \\ &= \int_0^t \nabla_{\Delta\bar{x}} G(s, x_s, 0)\sigma^{-1}\, d\bar{v}_s\end{aligned}$$

Using the fact that $\bar{v}_s$ is a Brownian motion with respect to P and that

$$\sup_{s\leqslant t, z\in R} |G(s, z, \bar{x})|, \qquad \sup_{s\leqslant t, z\in R} \left|\frac{\partial}{\partial s} G(s, z, \bar{x})\right|$$

converge to 0 as $\Delta\bar{x}\to 0$ (by the assumptions on F), one sees that for $\Delta\bar{x}$ sufficiently small, $dQ(\Delta\bar{x})/dP\in L_2(P)$ and, using uniform (in $s\leqslant t$ and $z\in R$) differentiability of $G(s, z, \Delta\bar{x})$, $(\partial/\partial s)G(s, z, \Delta\bar{x})$ in $\Delta\bar{x}$ at $\Delta\bar{x}=0$, that $dQ(\Delta\bar{x})/dP$ is differentiable in $\Delta\bar{x}$ at $\Delta\bar{x}=0$ in the sense of $L_2(P)$. Moreover,

$$\left\|\nabla_{\Delta\bar{x}} \frac{dQ(\Delta\bar{x})}{dP}\bigg|_{\Delta\bar{x}=0}\right\|_{L_2(P)} = \int_0^t \|\nabla_{\Delta\bar{x}} G(s, x_s, 0)\|_{L_2(P)}\, ds$$

(by the isometry property of the stochastic integral), which is bounded in $\bar{x}$, since $\nabla_{\Delta\bar{x}} G(s, z, 0)$ is bounded in $s\leqslant t$, $z\in R$. By elementary measure-theoretic reasoning, these properties are preserved under projection onto the t coordinate, i.e., $\delta_{\bar{x}+\Delta\bar{x}}P_t^e \ll \delta_{\bar{x}}P_t^e$ and $d(\delta_{\bar{x}+\Delta\bar{x}}P_t^e)/d(\delta_{\bar{x}}P_t^e)$ is differentiable in $\Delta\bar{x}$ at $\Delta\bar{x}=0$ in the sense of $L_2(\delta_{\bar{x}}P_t^e)$ and the $L_2(\delta_{\bar{x}}P_t^e)$ norm of the derivative is bounded in $\bar{x}$.

We now turn to the estimate of δ defined by (14). Let $f_t^e(\bar{x}, \cdot)$ denote the density of $\delta_{\bar{x}}P_t^e$ with respect to the Lebesgue measure existing by Proposition 5. For independent $\bar{X}_t^e$, $\bar{Y}_t^e$ starting in $\bar{x}, \bar{y}$, respectively ($\bar{x}$, $\bar{y}\in K$), we get the estimate

$$\delta(\bar{x}, \bar{y}, e) \geqslant p(\bar{x}, \bar{y}, e)\int_{\check{K}} \min(f_1^e(\bar{x}, \bar{z}), f_1^e(\bar{y}, \bar{z}))\, d\,\mathrm{Leb} \tag{18}$$

where

$$\check{K} := \{(v, x)\in E \mid |v| \leqslant (A+c)\beta^{-1}\} \qquad (\text{so } \check{K}^2 \subseteq K)$$

$$p(\bar{x}, \bar{y}, e) = \min(\delta_{\bar{x}}P_1^e(\check{K}), \delta_{\bar{y}}P_1^e(\check{K}))$$

We know that for all $\bar{x} \in E$, $f_1^e(\bar{x}, \bar{z}) > 0$ for almost all $\bar{z} \in E$ (with respect to the Lebesgue measure), since $\delta_{\bar{x}} P_t^e(D) > 0$ for each $D \subseteq E$ with positive Lebesgue measure. So the right-hand side of (18) is strictly positive for all $(\bar{x}, \bar{y})$, $|e| \leqslant A$. Furthermore, it depends continuously on $\bar{x}$, $\bar{y}$, and e by Proposition 5. Thus,

$$\check{\delta} = \inf_{\substack{\bar{x}, \bar{y} \in K \\ |e| \leqslant A}} \delta(\bar{x}, \bar{y}, e) > 0$$

Consequently,

$$\eta_t^e \leqslant \int_1^t \eta_{t-s}^e \, \hat{\sigma}(ds) + (\hat{\sigma}\{1\} - \check{\delta}) \eta_{t-1}^e \qquad (|e| \leqslant A)$$

We therefore see from the following lemma (using monotonicity of η_t^e) that for some $\check{\lambda}, \check{C} > 0$

$$\eta_t^e \leqslant \check{C} \exp(-\check{\lambda} t) \qquad (t \geqslant 0, |e| \leqslant A) \tag{19}$$

Lemma 8. Let m be a nonnegative measure on $[1, \infty)$ with $m([1, \infty)) < 1$ and $\int_{[1,\infty)} \exp(\gamma t) \, m(dt) < \infty$ for some $\gamma > 0$. Define η_t for $t \in R$ inductively on intervals $(n, n-1]$, $n \in N$, by

$$\eta_t = 1 \qquad \text{for} \quad t \in (-\infty, 0]$$

$$\eta_t = \int_{[1,\infty)} \eta_{t-s} \, m(ds)$$

(obviously, there exists exactly one function η_t satisfying this). Then for some $c, \kappa > 0$, $\eta_t \leqslant c \exp(-\kappa t)$ for all $t \in R$.

Proof of Lemma 8. Let $M := m([1, \infty))$, $D := \int_{[1,\infty)} \exp(\gamma t) \, m(dt)$. For each $t > 0$, $m([t, \infty)) \leqslant D \exp(-\gamma t)$, so m is stochastically smaller than the measure $\hat{m}$ on $[1, \infty)$ having as density with respect to the Lebesgue measure the function

$$\hat{f}(t) := \begin{cases} \gamma D \exp(-\gamma t) & \text{if} \quad t \geqslant t_0 := \gamma^{-1} \ln(DM^{-1}) \\ 0 & \text{if} \quad t < t_0 \end{cases}$$

[one easily verifies that $t_0 \geqslant 1$ and $\int_{t_0}^\infty \hat{f}(t) \, dt = M$]. Now define $\hat{\eta}_t$: $R \to R_+$ inductively by

$$\hat{\eta}_t = 1 \qquad \text{for} \quad t \in (-\infty, 0]$$

$$\hat{\eta}_t = \int_{[1,\infty)} \hat{\eta}_{t-s} \hat{m}(ds) = \int_{t_0}^\infty \hat{\eta}_{t-s} \hat{f}(s) \, ds$$

Using monotonicity of $\hat{\eta}_t$ and the fact that $\hat{m} \geqslant m$ in the stochastic ordering, one sees by induction on intervals $(n, n+1]$ that $\hat{\eta}_t \geqslant \eta_t$, so it suffices to establish the estimation $\hat{\eta}_t \leqslant c \exp(-\kappa t)$ for some $c, \kappa > 0$. We choose κ so that

$$\gamma D \int_{t_0}^{\infty} \exp[-(\gamma - \kappa)s]\, ds \leqslant 1$$

[which is possible since $\int_{t_0}^{\infty} \hat{f}(s)\, ds = M < 1$] and c so that $\hat{\eta}_s \leqslant c \exp(-\kappa s)$ in $[-\infty, 1]$. Then it follows by induction that

$$\begin{aligned}\hat{\eta}_t &\leqslant \int_{t_0}^{\infty} c \exp[-\kappa(t-s)] \cdot \gamma D \exp(-\gamma s)\, ds \\ &\leqslant c \exp(-\kappa t)\end{aligned}$$

for all $t \in R$.

We now complete the proof of Theorem 3. We show that for any $e \in R$ with $|e| < A$ and $(v, x) \in E$ with $|v| \leqslant v_0$ (and hence for any $e \in R$ and $(v, x) \in E$, since A and v_0 defined earlier can be taken arbitrarily large), $\delta_{(v,x)} P_t^e$ as a function of t is a Cauchy sequence (with respect to the variational norm). For any $t, s > 0$ we have, by arguments as used to establish (13),

$$\tfrac{1}{2} \| \delta_{\bar{x}} P_{t-s}^e - \delta_{\bar{x}} P_t^e \| \leqslant \int_{[0,t]} v^{(\bar{x},s,e)}(du)\, \eta_{t-u}^e$$

where $v^{(\bar{x},s,e)}$ is the distribution on $[0, \infty)$ of the stopping time

$$\tau^{(\bar{x},s,e)} := \{\inf u \geqslant 0 \mid (\bar{X}_u^e, \bar{Y}_u^e) \in K\}$$

$\bar{X}_u^e$, $\bar{Y}_u^e$ being two processes (4) coupled independently with initial distributions $\delta_{\bar{x}}$, $\delta_{\bar{x}} P_s^e$, respectively. From Proposition 4 we see that there is a probability measure $\hat{v}$ on $[0, \infty)$ which is stochastically larger than $v^{(\bar{x},s,e)}$ for any $s \geqslant 0$, $\bar{x} = (v, x) \in E$ with $|v| \leqslant v_0$, $e \in R$ with $|e| \leqslant A$, and which satisfies $\hat{v}([1, \infty)) \leqslant C \exp(-\tilde{\lambda} t)$ for some $\tilde{C}, \tilde{\lambda} \geqslant 0$. So we have for all $t, s \geqslant 0$, using (19),

$$\begin{aligned}\tfrac{1}{2} \| \delta_{\bar{x}} P_{t-s}^e - \delta_{\bar{x}} P_t^e \| &\leqslant \int_{[0,t]} \hat{v}(du)\, \eta_{t-u}^e \\ &\leqslant \eta_{t/2}^e + \hat{v}([t/2, \infty)) \\ &\leqslant C \exp(-\lambda t) \qquad (20)\end{aligned}$$

with $C := \max(\check{C}, \tilde{C})$ and $\lambda := \inf(\check{\lambda}, \tilde{\lambda}) \cdot 2^{-1}$. So $\delta_{\bar{x}} P_t^e$ is indeed a Cauchy sequence. Consequently, the corresponding densities $f_t^e(\bar{x}, \cdot)$ existing by Proposition 4 have a limit f^e in $L_1(E)$ (with respect to the Lebesgue measure) which does not depend on $\bar{x}$ by (19). Let π_e be the probability measure on E associated to the density f^e. Relation (20) shows that

$$\tfrac{1}{2} \|\delta_{\bar{x}} P_t^e - \pi_e\| \leqslant C \exp(-\lambda t) \tag{21}$$

for $\bar{x} = (v, x) \in E$ with $|v| \leqslant v_0$, $|e| \leqslant A$, and it easily follows that π_e is invariant with respect to P_t^e. By noting that $\frac{1}{2} \|\delta_{\bar{x}} P_t^e - \pi_e\| < 1$ for all $t > 0$, which follows from Lemma 5 and almost sure positivity of $f_t^e(\bar{x}, \cdot)$, we see that

$$\tfrac{1}{2} \|\delta_{\bar{x}} P_t^e - \pi_e\| \leqslant q^t \qquad (|e| \leqslant A,\ |V(\bar{x})| \leqslant v_0)$$

for some $q < 1$. This finishes the proof of Theorem 3.

3. CALCULATION OF THE MOBILITY

In order to establish the differentiability of π_e and $\bar{v}_e$ in e and to calculate the mobility, I will now make precise the perturbation argument sketched in the introduction.

Instead of working with L_e^*, I rewrite and solve Eqs. (8)–(10) in a weak form so one does not have to care about the smoothness properties of the involved functions. I use $C_0^2(E) :=$ space of twice continuously differentiable functions φ: $E \to R$ such that $\varphi(v, x) \to 0$ if $|v| \to \infty$ as the space of test functions and write $\langle f, \varphi \rangle$ for $\int_E f\varphi \, d\,\text{Leb}$, f: $E \to R$.

The next theorem gives an expression which presents a weak form of (6). (In particular, in view of Propositions 13 and 17, this shows that π_e is differentiable in e in a weak sense.)

Theorem 9. For each $e \in R$,

$$\int_E \varphi \, d\tilde{\pi}_e = \langle f_0, \varphi \rangle + e \langle F, \varphi \rangle + e^2 \int_0^\infty \langle H, P_t^e \varphi \rangle \, dt \qquad [\varphi \in C_0^2(E)] \tag{22}$$

with

$$F\colon\ E \to R, \qquad F(v, x) = -\frac{2\beta}{\sigma^2} f_0(v, x)\, G(-v, x)$$

$$G\colon\ E \to R, \qquad G(\bar{x}) = \int_0^\infty P_t V(\bar{x}) \, dt$$

$$H\colon\ E \to R, \qquad H(v, x) = \frac{\partial}{\partial v} F(v, x)$$

defines a finite measure $\tilde{\pi}_e$ on E which is invariant with respect to P_t^e.

Proof. I first show that the expressions occurring in Theorem 9 are well defined.

Proposition 10. The function G is well defined and differentiable. For any $\kappa > 0$, there exists $D > 0$ such that for all $\bar{x} = (v, x) \in E$, $|\nabla_{\bar{x}} G(\bar{x})| \leqslant D \exp(\kappa |v|)$.

I first show the following result.

Lemma 11. For any $\kappa > 0$, there exist $\bar{\kappa} > 0$ and $C > 0$ such that $|P_t V(v, x)| \leqslant C \exp(\kappa |v|) \exp(-\bar{\kappa} t)$ for all $t > 0$, $(v, x) \in E$.

Proof. Choose $v_0 > c := \sup\{|\phi'(x)| \mid |x \in R\}$. By Theorem 3, there exists $q < 1$ such that for all $(v, x) \in E$ with $|v| \leqslant v_0$ and $t > 0$

$$\tfrac{1}{2} \|\delta_{\bar{x}} P_t - \pi\| \leqslant q^t \qquad (\text{with } \pi := \pi_0) \tag{23}$$

Consider the first entrance time into $K := \{(v, x) \in E \mid |v| \leqslant v_0\}$:

$$\tau := \inf\{t \geqslant 0 \mid \bar{X}_t \in K\}$$

By Proposition 4, for $0 < \lambda < (\beta v_0 - c)^2 (2\sigma^2)^{-1}$, there exists $\psi(\lambda) < \infty$ such that

$$E_{(v,x)} \exp(\lambda t) \leqslant \exp[\psi(\lambda)(|v| - v_0)] \tag{24}$$

By the dominated convergence theorem, we can assume that $\psi(\lambda) \to 0$ as $\lambda \to 0$. By Tchebychev's inequality, (24) implies

$$P_{(v,x)}\{\tau > s\} \leqslant \exp[\psi(\lambda)(|v| - v_0)] \exp(-\lambda s), \qquad s > 0,\ (v, x) \in E \tag{25}$$

For any $(v, x) \in E$, if μ denotes the distribution of τ on R_+ under the condition $\bar{X}_0 = (v, x)$, we have, for $t > 0$, using (23) and (25),

$$\begin{aligned} &\frac{1}{2} \|\delta_{(v,x)} P_t - \pi\| \\ &\leqslant \int_0^t \mu(ds) q^{t-s} + P_{(v,x)}\{\tau > t\} \\ &\leqslant P_{(v,x)}\left\{\tau \leqslant \frac{t}{2}\right\} q^{t/2} + P_{(v,x)}\{\tau > t\} \\ &\leqslant q^{t/2} + \exp[\psi(\lambda)(|v| - v_0)] \exp(-\lambda t/2) \end{aligned} \tag{26}$$

We further have, for any $l > 0$, $t > 0$, and $(v, x) \in E$,

$$|P_t V(v, x)| \leqslant |P_t V^l(v, x)| + |P_t(V - V^l)(v, x)| \tag{27}$$

with

$$V^l\colon\quad E \to R, \qquad V^l(v, x) = \begin{cases} v & \text{if} \quad |v| \leqslant l \\ 0 & \text{otherwise} \end{cases}$$

and

$$\begin{aligned} &|P_t V^l(v, x)| \\ &\quad \leqslant l \|\delta_{(v,x)} P_t\| \\ &\quad \leqslant 2lq^{t/2} + 2l \exp[\psi(\lambda)(|v| - v_0)] \exp(-\lambda t/2) \qquad \text{(by (26)]} \end{aligned} \tag{28}$$

$$\begin{aligned} &|P_t(V - V^l)(v, x)| \\ &\quad \leqslant \int_{\max(v,l)}^{\infty} u\, v_v(du) \qquad [\text{by Proposition 4(a) if } |v| > c \\ &\qquad\qquad\qquad\qquad\qquad (\text{in particular, if } |v| > v_0)] \\ &\quad \leqslant 2(\beta v - c)\sigma^{-2} \int_{\max(v,l)}^{\infty} \exp[-2(\beta v - c)\, \sigma^{-2}(u - v)]\, u\, du \\ &\quad \leqslant (\max(v, l) + \sigma^2[2(\beta v - c)]^{-1}) \\ &\qquad \cdot \exp[-2(\beta v - c)\sigma^{-2} \max(v, l)] \end{aligned} \tag{29}$$

For $|v| \leqslant v_0$, we have, instead of (29),

$$\begin{aligned} &|P_t(V - V^l)(v, x)| \\ &\quad \leqslant \int_{\max(v_0,l)}^{\infty} u_{v_{v_0}}(du) \\ &\quad (\max(v_0, l) + \sigma^2[2(\beta v_0 - c)]^{-1}) \\ &\quad \cdot \exp[-2(\beta v_0 - c)\sigma^{-2} \max(v_0, l)] \end{aligned} \tag{30}$$

Returning to the statement of Lemma 11, let $\kappa > 0$ be given. Choose $\lambda > 0$ such that $\lambda < (\beta v_0 - c)^2 (2\sigma^2)^{-1}$ [to make (24) true] and $\psi(\lambda) < \kappa$. Now taking $\bar{\kappa} < \min(\lambda/2,\ -(\ln q)/2,\ 2(\beta v_0 - c)\sigma^{-2})$ and setting $l = t$ in (28)–(30), we see that in each of the inequality chains (28)–(30), the last expression is $\leqslant C \exp(\kappa |v|) \exp(-\bar{\kappa} t)$ for some $C > 0$; so it follows by (27) that for some $C > 0$,

$$|P_t V(v, x)| \leqslant C \exp(\kappa |v|) \exp(-\bar{\kappa} t)$$

for all $(v, x) \in E,\ t > 0$.

I now continue the proof of Proposition 10.

From Lemma 11 we see that it makes sense to define the function G pointwise by

$$G(\bar{x}) := \int_0^\infty P_t V(\bar{x})\, dt$$

We further see, using Proposition 4(a)(ii), that for each $\bar{x} \in E$ and $t > 0$, $P_s V \in L_2(\delta_{\bar{x}} P_t)$ for all $s \geqslant 0$ and that for each $\kappa > 0$, there exist $\bar{\kappa} > 0$, $C' > 0$ such that

$$\|P_s V\|_{L_2(\delta_{(v,x)} P_t)} \leqslant C' \exp(\kappa |v|) \exp(-\bar{\kappa} s)$$

so for all $t > 0$, $\int_t^\infty P_s V\, ds$ exists in $L_2(\delta_{(v,x)} P_t)$ and has $L_2(\delta_{(v,x)} P_t)$-norm $\leqslant C' \bar{\kappa}^{-1} \exp(\kappa |v|)$. By Proposition 5, the density of $\delta_{\bar{y}} P_t$ with respect to $\delta_{\bar{x}} P_t$ is differentiable in $\bar{y}$ at $\bar{y} = \bar{x}$ in $L_2(\delta_{\bar{x}} P_t)$ and the derivative has $L_2(\delta_{\bar{x}} P_t)$-norm bounded in $\bar{x}$. By Hölder's inequality (applied to the probability measure $\delta_{\bar{x}} P_t$) it follows that for $t > 0$,

$$\int_t^\infty P_s V(\bar{x})\, ds = \int_0^\infty \int_E P_s V(\bar{z})\, d(\delta_{\bar{x}} P_t)(\bar{z})\, ds$$

is differentiable in $\bar{x}$ and for any $\kappa > 0$, there exists $C'' > 0$ such that

$$\left| \nabla_{(v,x)} \left(\int_t^\infty P_s V(v,x)\, ds \right) \right| \leqslant C'' \exp(\kappa |v|), \qquad (v,x) \in E$$

The proof of Proposition 10 is therefore finished by the following claim:

Lemma 12. For each $t > 0$, $\int_0^t P_s V(\bar{x})\, ds$ is differentiable in $\bar{x}$. The derivative is bounded in $\bar{x}$.

Proof. Consider, e.g., the norm in R^2,

$$\|(v,x)\| = |v| + |x|$$

Let $\delta(\bar{x}) \in R^2$ for $\bar{x} \in R^2$ denote the drift of the process (2) in $\bar{x}$. Then obviously

$$\|\delta(\bar{x}) - \delta(\bar{y})\| \leqslant [1 + \beta + \max_{x \in R} |\Phi''(x)|]\, \|\bar{x} - \bar{y}\|, \qquad \bar{x}, \bar{y} \in R^2$$

For any $\bar{x}, \Delta\bar{x} \in R^2$, we can therefore choose a coupling of two processes (2) $\bar{X}_t$ and $\bar{X}'_t$ starting in $\bar{x}$, $\bar{x} + \Delta\bar{x}$, respectively, on a probability space Ω such that for all $s > 0$, $\omega \in \Omega$,

$$\|\bar{X}_s(\omega) - \bar{X}'_s(\omega)\| \leqslant \|\Delta\bar{x}\| \exp[(1 + \beta + \max|\Phi''|)t]$$

This shows that for any $\bar{x}$, $\Delta\bar{x} \in R^2$, $s > 0$,

$$|P_s V(\bar{x} + \Delta\bar{x}) - P_s V(\bar{x})| \leqslant \|\Delta\bar{x}\| \exp[(1 + \beta + \max|\Phi''|)t] \tag{31}$$

As a consequence of Proposition 5, we know that $P_s V$ is differentiable for each $s > 0$ [since $V \in L_2(\delta_{\bar{x}} P_s)$ for any $\bar{x} \in R^2$ by Proposition 4]. Relation (31) shows that $\nabla_{\bar{x}} P_s V(\bar{x})$ is bounded in $\bar{x} \in E$ and $0 \leqslant s \leqslant t$, t fixed. This completes the proof of Proposition 10.

Proposition 13. For all $A > 0$, there exist C, $\rho > 0$ such that for each $\varphi \in C_0^2(E)$ and $t > 0$,

$$|\langle H, P_t^e \varphi \rangle| \leqslant \sup_{\bar{x} \in E} |\varphi(\bar{x})| \, C \exp(-\rho t)$$

for all $e \in R$ with $|e| \leqslant A$.

Proof. By Proposition 10, for some $\tilde{C}_1$, $\tilde{C}_2 > 0$,

$$|G(v, x)| \leqslant \tilde{C}_1 \exp(\tilde{C}_2 |v|), \qquad (v, x) \in E \tag{32}$$

Consequently, for each $\kappa > 0$, there exists $\tilde{\tilde{C}} > 0$ such that

$$|H(v, x)| \leqslant \tilde{\tilde{C}} \exp(-\kappa |v|), \qquad (v, x) \in E \tag{33}$$

Let $H^+ := \max(H, 0)$, $H^- := \max(-H, 0)$. As a consequence of (32),

$$\int_E H^+ \, d\,\mathrm{Leb} = \int_E H^- d\,\mathrm{Leb} := m$$

Given $A > 0$, take $v_0 > (c + A)\beta^{-1}$. Define for $|e| < A$ the stopping time

$$\tau_e := \inf\{t \geqslant 0 \mid |V_t^e| \leqslant v_0\}$$

By Proposition 4(a), there exist λ, $\lambda' > 0$ such that for all e with $|e| < A$ and all $(v, x) \in E$ with $|v| \geqslant v_0$,

$$E_{(v,x)} \exp(\lambda \tau_e) \leqslant \exp[\lambda'(|v| - v_0)]$$

It follows with (33) that

$$B := \sup_{\substack{e \in R \\ |e| \leqslant A}} \mathrm{m} \cdot E_{m^{-1} H^+ \mathrm{Leb}} \exp(\lambda \tau_e) < \infty \tag{34}$$

By Theorem 3, there exists $q < 1$ such that for all $(v, x) \in E$ with $|v| \leqslant v_0$ and for all $e \in R$ with $|e| \leqslant A$

$$\|\delta_{(v,x)} P_t^e - \pi_e\| \leqslant 2q^t \qquad (t \geqslant 0) \tag{35}$$

Let μ_e be the distribution of τ_e on R_+ for the initial distribution of $\bar{X}_t^e$ H^+m^{-1} Leb. Then

$$\begin{aligned}
\|(H^+m^{-1}\text{ Leb})P_t^e - \pi_e\| &\leqslant 2\int_0^t \mu_e(ds)q^{t-s} + 2\mu((t, \infty)) \qquad \text{[by (35)]}\\
&\leqslant 2([0, t/2])q^{t/2} + 2\mu((t/2, \infty))\\
&\leqslant 2q^{t/2} + 2B\exp(-\lambda t/2) \qquad \text{[by (34)]}\\
&\leqslant (C/2m)\exp(-\rho t) \qquad \text{for some } C, \rho > 0 \qquad (36)
\end{aligned}$$

In the same way, one has for H^-

$$\|(m^{-1}H^-\text{ Leb})P_t^e - \pi_e\| \leqslant (C/2m)\exp(-\rho t)$$

This and (36) imply

$$\begin{aligned}
\|(m^{-1}H\text{ Leb})P_t^e\| &= \|(m^{-1}H^+\text{ Leb})P_t^e - (m^{-1}H^-\text{ Leb})P_t^e\|\\
&\leqslant Cm^{-1}\exp(-\rho t) \qquad (37)
\end{aligned}$$

Hence for all $\varphi \in C_0^2(E)$

$$\begin{aligned}
|\langle H, P_t^e\varphi\rangle| &\leqslant \sup_{\bar{x}\in E} |\varphi(x)| \cdot m\|(m^{-1}H\text{ Leb})P_t^e\|\\
&\leqslant \sup |\varphi(x)| \cdot C\exp(-\rho t)
\end{aligned}$$

From Proposition 13 and the integrability of f_0, F with respect to the Lebesgue measure, we see that $\tilde{\pi}_e$ is well defined and that for some $d > 0$

$$\int_E \varphi\, d\pi_e \leqslant d \cdot \sup_{\bar{x}\in E} |\varphi(\bar{x})|, \qquad \varphi \in C_0^2(E) \qquad (38)$$

By letting $\varphi \to 1$ in a suitable way, it follows that $\tilde{\pi}_e$ is a finite measure on E.

I now show that for each $e \in R$, $\tilde{\pi}_e$ is invariant with respect to P_t^e. The proof uses the following two lemmas.

Lemma 14. For any $e \in R$, a finite measure μ on E is invariant with respect to P_t^e iff $\int_E L_e\varphi\, d\mu = 0$ for all $\varphi \in C_0^2(E)$.

Lemma 15. For each $\psi \in C_0^2(E)$,

$$\langle f_0 P_t V, \hat{\psi}\rangle = \langle \widehat{f_0 V}, P_t\psi\rangle, \qquad s, t > 0,$$
$$\hat{\varphi}(v, x) = \varphi(-v, x) \quad \text{for} \quad \varphi\colon E \to R$$

Lemmas 14 and 15 follow from the following lemma. For the proof of Lemma 15 one uses the reversibility property

$$\langle f_0 L\phi, \hat{\psi}\rangle = \langle \widehat{f_0\phi}, L\psi\rangle$$

for functions ϕ which are twice continuously differentiable and ϕ, $L\phi \in L_1(\pi_0)$, which can be checked by an easy calculation.

Lemma 16. For $\varphi \in C_0^2(E)$ or $\varphi \equiv V$, $\int_0^t P_s\varphi\, ds$ is well defined and twice continuously differentiable and

$$P_t\varphi - \varphi = L \int_0^t P_s\varphi\, ds, \qquad t \geqslant 0$$

For $\varphi \in C_0^2(E)$, $\int_0^t P_s\varphi\, ds \in C_0^2(E)$.

Proof. By similar arguments as used in the proof of Proposition 10, using Proposition 5.

Since f_0 is the invariant density of $\bar{X}_t^0$, we know by Lemma 14 that

$$\langle f_0, L_0\varphi\rangle = 0, \qquad \varphi \in C_0^2(E) \tag{39}$$

[this corresponds to Eq. (8)].

We next show

$$\langle F, L_0\varphi\rangle = -\langle f_0, \nabla_v\varphi\rangle, \qquad \varphi \in C_0^2(E) \tag{40}$$

[which corresponds to Eq. (9)]: For any function $h\colon E \to R$, by $\hat{h}$ we denote the function $E \to R$ defined by $\hat{h}(v, x) = h(-v, x)$. Then for all $\varphi \in C_0^2(E)$

$$\langle F, L_0\varphi\rangle = \lim_{s\to 0} s^{-1}\langle F, P_s\varphi - \varphi\rangle$$

[since, as a consequence of Lemma 11, $F = -2\beta\sigma^{-2} f_0 \int_0^\infty \widehat{P_t V}\, dt \in L_1(E, \text{Leb})$ and since $\lim_{s\to 0} s^{-1}(P_s\varphi - \varphi) \to L_0\varphi$ uniformly]

$$= \lim_{s\to 0} -s^{-1} 2\beta\sigma^{-2} \int_0^\infty (f_0 \widehat{P_t V}, P_s\varphi\rangle - \langle f_0 \widehat{P_t V}, \varphi\rangle\, dt$$

[by Lemma 11]

$$= \lim_{s\to 0} -s^{-1} 2\beta\sigma^{-2} \int_0^s \langle f_0 V, P_t\varphi\rangle\, dt$$

[by Lemma 15]

$$= -2\beta\sigma^{-2}\langle f_0 V, \varphi\rangle$$

[since $\lim_{t\to 0} P_t\varphi = \varphi$ uniformly]

$$= \langle \nabla_v f_0, \varphi\rangle$$

$$= -\langle f_0, \nabla_v\varphi\rangle$$

[by partial integration]

We finally show that

$$\int_0^\infty \langle H, P_t^e L_e \varphi \rangle \, dt = -\langle F, \nabla_v \varphi \rangle, \qquad e \in R, \quad \varphi \in C_0^2(E) \tag{41}$$

[which corresponds to Eq. (10)]:

$$\int_0^\infty \langle H, P_t^e L_e \varphi \rangle \, dt = \int_0^\infty \langle H, P_t^e \lim_{s \to 0} s^{-1}(P_s^e \varphi - \varphi) \rangle \, dt$$

$$= \lim_{s \to 0} s^{-1} \int_0^\infty \langle H, P_{t-s}^e \varphi - P_t^e \varphi \rangle \, dt$$

[by Proposition 13, since $\lim_{s \to 0} s^{-1}(P_s^e \varphi - \varphi) = L_e \varphi$ uniformly]

$$= \lim_{s \to 0} s^{-1} \int_0^s \langle H, P_t^e \varphi \rangle \, dt$$

$$= \langle H, \varphi \rangle$$

[since $\lim_{s \to 0} P_t \varphi = \varphi$ uniformly]

$$= -\langle F, \nabla_v \varphi \rangle$$

[by partial integration]

Since $L_e = L_0 + e\nabla_v$, (39)–(41) imply $\int_E L_e \varphi \, d\tilde{\pi}_e = 0$ for all $\varphi \in C_0^2(E)$, so $\tilde{\pi}_e$ is invariant with respect to P_t^e as was claimed.

Proposition 17. The function $\gamma\colon R \to R$, $\gamma(e) = \tilde{\pi}_e(E)$ is differentiable at $e = 0$ with $\gamma(0) = 1$, $\gamma'(0) = 0$.

Proof. $\gamma(0) = 1$ is obvious. From Proposition 13 and the definition (22) of $\tilde{\pi}_e$ it follows that γ is differentiable at $e = 0$ with derivative $\int_E F \, d\,\text{Leb}$. One easily calculates that $\int_E F \, d\,\text{Leb} = -2\beta\sigma^{-2} \int_E G \, d\pi_0 = 0$.

Since for each $e \in R$ there exists exactly one invariant probability measure for $\bar{X}_t^e$, namely π_e (as shown in Section 2), Theorem 9 and Proposition 17 show that for $|e|$ small enough

$$\pi_e = \gamma^{-1}(e) \tilde{\pi}_e$$

We now finish the proof of Theorem 2.

By Proposition 4(a), for $v_0 > (c + |e|)\beta^{-1}$, we have $\delta_{(x,0)} P_t^e \leqslant m_{v_0}$ for all $t>0$. By (21), this estimate carries over to π_e. So the mean velocity $\bar{v}_e=\int_E V\,d\pi_e$ is well defined.

Lemma 18. For $e\in R$ let μ_e be the finite measure on E defined by

$$\int_E \varphi\,d\mu_e=\int_0^\infty \langle H, P_t^e\varphi\rangle\,dt, \qquad \varphi\in C_0^2(E)$$

Then $V\in L_1(\mu_e)$ for all $e\in R$.

Proof. As a consequence of (33) and Proposition 4(a)(ii), for each $e\in R$, there exist D, $\sigma>0$ such that for all $t>0$, $l>0$,

$$H\cdot\text{Leb}\,P_t^e(\{v,x)\in E\mid |v|>l\})\leqslant D\exp(-\sigma l)$$

$\{$I write $H\cdot\text{Leb}\,P_t^e$ for $m[(m^{-1}H\,\text{Leb})P_t^e]$, $m=\int|H|\,d\text{Leb}.\}$ It follows with (37) in Proposition 13 that for some C, $\rho>0$

$$\begin{aligned}&\mu_e(\{(v,x)\in E\mid |v|>l\})\\ &\qquad\leqslant\int_l^\infty \|(H\,\text{Leb})P_t^e\|+lD\exp(-\sigma l)\\ &\qquad\leqslant C\rho^{-1}+lD\exp(-\sigma l)\end{aligned}$$

which is exponentially decreasing in l. Hence $V\in L_1(\mu_e)$. Lemma 18 shows that the function $\tilde{v}$: $R\to R$, $\tilde{v}(e)=\int V\,d\tilde{\pi}_e$ is well defined and differentiable at $e=0$ with derivative $\tilde{v}'(0)=\langle F, V\rangle$. So by Proposition 17, $\bar{v}_e=\int V\,d\pi_e$ is differentiable at $e=0$ with the same derivative, i.e.,

$$\left.\frac{d}{de}\bar{v}_e\right|_{e=0}=\langle F, V\rangle=2\beta\sigma^{-2}\int_E VG\,d\pi$$

as was claimed in Theorem 2.

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